Year 12 Semester 2 Examination, 2017

Question/Answer Booklet

Hale School

MATHEMATICS SPECIALIST

Section One
Calculator-free

_____

Student Name

Teacher: Mr Hill Mr Bausor
(circle)

Score: ____ (out of 51)

Time allowed for this section
Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	51	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2017*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **(51 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

Given $z = \cos\alpha + i\sin\alpha$ and $w = \cos\beta + i\sin\beta$, where $w \neq 0$, establish the following identities:

(a) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ (2 marks)

$$LHS = \left| \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} \times \frac{\cos\beta - i\sin\beta}{\cos\beta - \sin\beta} \right|$$

$$= \left| \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta + i(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{\cos^2\beta + \sin^2\beta} \right|$$

$$= \sqrt{(\cos\alpha\cos\beta + i\sin\alpha\sin\beta)^2 + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)^2}$$

$$= 1$$

$$= RHS$$

(b) $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$ (2 marks)

$$LHS = \arg(\cos\alpha\cos\beta + \sin\alpha\sin\beta + i(\sin\alpha\cos\beta - \cos\alpha\sin\beta))$$

$$= \arg(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

$$= \alpha + \beta$$

$$= \arg(\cos\alpha + i\sin\alpha) + \arg(\cos\beta + i\sin\beta)$$

$$= RHS$$

Question 2 **(8 marks)**

Let a system of equations be defined as

$$\begin{aligned} x - y &= p \\ x + qy - 3z &= 7 \\ 4x - y - 3z &= 4 \end{aligned}$$

where of p and q are real constants.

(a) Determine the possible values of p and q if the system of equations show below has:

(i) a unique solution, (4 marks)

$$\begin{bmatrix} z & x & y & k \\ -3 & 1 & q & 7 \\ -3 & 4 & -1 & 4 \\ 0 & 1 & -1 & p \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & q & 7 \\ 0 & -3 & q+1 & 3 \\ 0 & 1 & -1 & p \end{bmatrix} R_1 - R_2$$

$$\begin{bmatrix} -3 & 1 & q & 7 \\ 0 & -3 & q+1 & 3 \\ 0 & 0 & q-2 & 3+3p \end{bmatrix} R_2 + 3R_3$$

$q - 2 = 3 + 3p$

$q \neq 2 \quad p \neq -1$

(ii) infinite solutions. (2 marks)

$q = 2 \quad p = 1$

(b) If $p = q = 2$ give a geometric interpretation of the system of equations. (2 marks)

No Soln

Intersecting pairs of planes form parallel lines

Question 3 **(6 marks)**

Find the equation of the tangent to the curve $e^y + y\ln x = 3$ at the point P, whose co-ordinates are $(1, \ln 3)$.

$$\frac{dy}{dx}e^y + \frac{dy}{dx}\ln x + \frac{y}{x} = 0 \qquad (1, \ln 3)$$

$$\frac{dy}{dx}e^{\ln 3} + \frac{dy}{dx}\ln 1 + \frac{\ln 3}{1} = 0$$

$$3\frac{dy}{dx} = -\ln 3$$

$$\frac{dy}{dx} = \frac{-\ln 3}{3}$$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - \ln 3 = \frac{-\ln 3}{3}(x - 1)$$

$$y = \frac{-\ln 3}{3}x + \frac{\ln 3}{3} + \ln 3$$

$$3y = 4\ln 3 - \ln 3 x$$

Question 4 **(12 marks)**

Let $g(x)=1+\frac{1}{x}$, $h(x)=(x+1)^2-4$ and $f[g(x)]=x+1$.

(a) Find the domain of $g(x)$. (1 mark)

$D_x = \{ x \neq 0 : x \in \mathbb{R} \}$

(b) State a suitable restriction on the domain of $h(x)$ for an inverse to exist and determine $h^{-1}(x)$ for this restriction. (4 marks)

Restrict domain $h(x)$ $x \geq -1$

$x = (y+1)^2 - 4$

$x + 4 = (y+1)^2$

$y = \pm\sqrt{x+4} - 1$

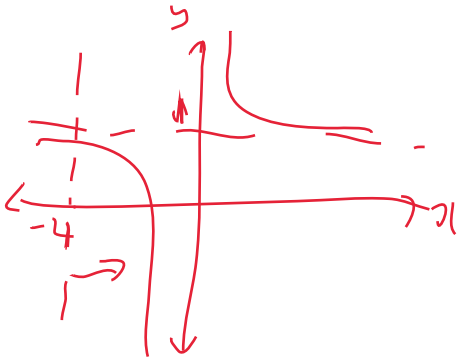
$h^{-1}(x) = \pm\sqrt{x+4} - 1$

$h^{-1}(x) = \sqrt{x+4} - 1$ for given restriction

(c) Find the domain and range of $g[h(x)]$. (4 marks)

$D = \{ x \neq 1, x \neq -3 : x \in \mathbb{R} \}$

$R = \{ y \leq \frac{3}{4} \cup y \geq 1 : y \in \mathbb{R} \}$



Question 4 continued…

(d) Determine $f(x)$. (3 marks)

$$g(x) = 1 + \frac{1}{x}$$

$$x = 1 + \frac{1}{y}$$

$$x - 1 = \frac{1}{y}$$

$$y = \frac{1}{x-1}$$

$$f(g(x)) = x + 1$$

$$f\left(g\left(g^{-1}(x)\right)\right) = \left(\frac{1}{x-1}\right) + 1$$

$$f(x) = \frac{1}{x-1} + 1$$

Question 5 **(8 marks)**

Let $\frac{dy}{dx} = y^2 - 4y - 5$ and $y_o = y(0) = 0$.

(a) Solve the differential equation above giving x as a function of y. (6 marks)

$$\frac{dy}{dx} = y^2 - 4y - 5$$

$$\int \frac{1}{y^2 - 4y - 5}\, dy = \int 1\, dx$$

$$\int \frac{1}{(y-5)(y+1)}\, dy = \int 1\, dx$$

$$\int \frac{\frac{1}{6}}{y-5} - \frac{\frac{1}{6}}{y+1}\, dy = \int 1\, dx$$

$$\ln|y-5| - \ln(y+1) = 6x + C$$

$$\ln\left|\frac{y-5}{y+1}\right| = 6x + C$$

$$\ln 5 = C$$

$$\therefore x = \frac{1}{6}\left(\ln\left|\frac{y-5}{y+1}\right| - \ln 5\right)$$

$$\frac{A}{y-5} + \frac{B}{y+1} = \frac{1}{(y-5)(y+1)}$$

$$A(y+1) + B(y-5) = 1$$

$y = -1$: $-6B = 1$, $B = -\frac{1}{6}$

$y = 5$: $6A = 1$, $A = \frac{1}{6}$

$(0, 0)$

(b) Apply Euler's incremental formula to find y_1, the next y-value from y_0, using a step size $\delta x = 0.1$. (2 marks)

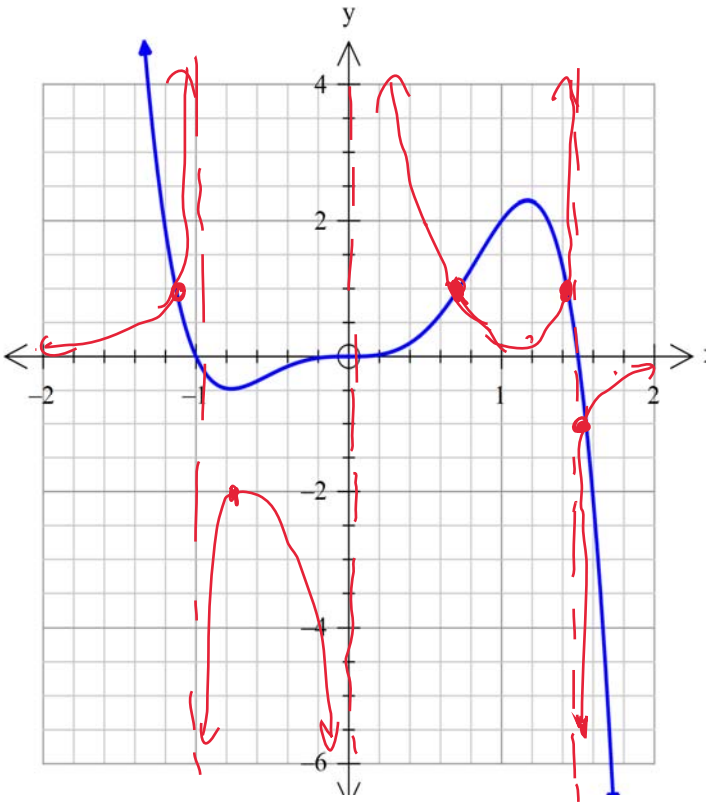
$$y_1 = y_0 + \frac{dy}{dx}\delta x$$

$$= 0 + (0 + 0 - 5)0.1$$

$$= -0.5$$

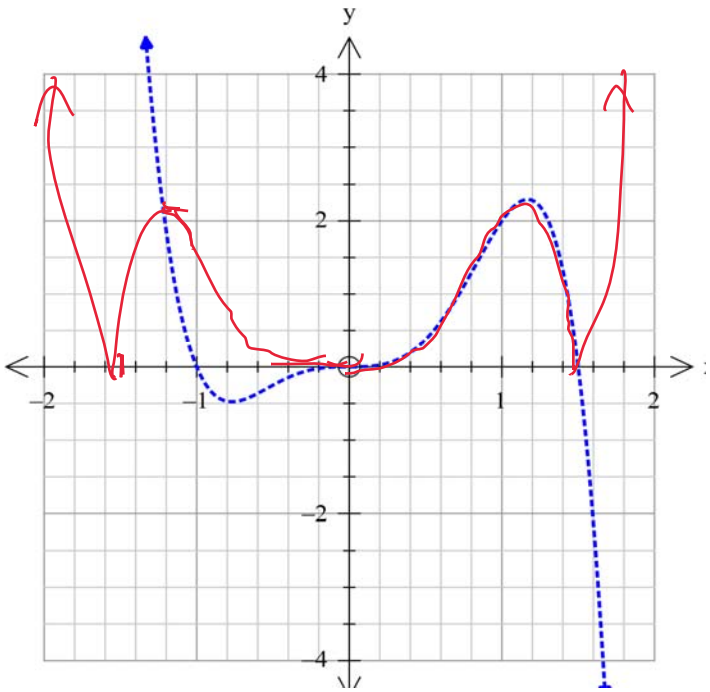
Question 6 **(6 marks)**

Given the graph of $y = f(x)$ is given as follows;



(a) On the axis above sketch the graph of $y = \dfrac{1}{f(x)}$. (3 marks)

(b) On the axis below, sketch the graph of $y = \left| f\left(|x|\right) \right|$. (3 marks)



Question 7 **(7 marks)**

Evaluate the following integral using the substitution provided.

$$\int_0^{\frac{3}{2}} \sqrt{9-x^2}\, dx \qquad \text{let } x = 3\cos\theta$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{9-9\cos^2\theta}\,.\,-3\sin\theta\, d\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta$$

$x = 0 \quad \theta = \frac{\pi}{2}$

$x = \frac{3}{2} \quad \theta = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 3\sin\theta\,.\,-3\sin\theta\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -9\sin^2\theta\, d\theta$$

$$\left[\cos 2x = 1 - 2\sin^2\theta\right]$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{9}{2}\left[\cos 2\theta - 1\right] d\theta$$

$$= \frac{9}{2}\left\{ \frac{\sin 2\theta}{2} - x \right\}_{\frac{\pi}{2}}^{\frac{\pi}{3}}$$

$$= \frac{9}{2}\left(\left(\frac{\sin\frac{2\pi}{3}}{2} - \frac{\pi}{3} \right) - \left(\frac{\sin\pi}{2} - \frac{\pi}{2} \right) \right)$$

$$= \frac{9}{2}\left(\frac{\sqrt{3}}{4} - \frac{\pi}{3} + \frac{\pi}{2} \right)$$

$$= \frac{9}{2}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6} \right)$$

$$= \frac{9}{24}\left(3\sqrt{3} + 2\pi\right)$$

END OF SECTION 1

Additional working space

Question number: _____

Additional working space

Question number: _____